

FINITE-TIME BLOWUP FOR THE INVISCID PRIMITIVE EQUATIONS OF OCEANIC AND ATMOSPHERIC DYNAMICS

CHONGSHENG CAO, SLIM IBRAHIM, KENJI NAKANISHI, AND EDRISS S. TITI

ABSTRACT. In an earlier work we have shown the global (for all initial data and all time) well-posedness of strong solutions to the three-dimensional viscous primitive equations of large scale oceanic and atmospheric dynamics. In this paper we show that for certain class of initial data the corresponding smooth solutions of the inviscid (non-viscous) primitive equations blow up in finite time. Specifically, we consider the three-dimensional inviscid primitive equations in a three-dimensional infinite horizontal channel, subject to periodic boundary conditions in the horizontal directions, and with no-normal flow boundary conditions on the solid, top and bottom, boundaries. For certain class of initial data we reduce this system into the two-dimensional system of primitive equations in an infinite horizontal strip with the same type of boundary conditions; and then show that for specific sub-class of initial data the corresponding smooth solutions of the reduced inviscid two-dimensional system develop singularities in finite time.

MSC Subject Classifications: 35Q35, 65M70, 86-08, 86A10.

Keywords: Blowup, Primitive equations, hydrostatic balance, Boussinesq equations, Euler equations, Navier–Stokes equations.

1. INTRODUCTION

The three-dimensional primitive equations for large scale oceanic and atmospheric dynamics are given by the system of partial differential equations:

$$u_t + uu_x + vu_y + wu_z + p_x - Rv = \nu_H \Delta_H u + \nu_3 u_{zz}, \quad (1)$$

$$v_t + uv_x + vv_y + wv_z + p_y + Ru = \nu_H \Delta_H v + \nu_3 v_{zz}, \quad (2)$$

$$p_z + T = 0, \quad (3)$$

$$T_t + uT_x + vT_y + wT_z = Q + \kappa_H \Delta_H T + \kappa_3 T_{zz}, \quad (4)$$

$$u_x + v_y + w_z = 0, \quad (5)$$

which are supplemented with the initial value (u_0, v_0, T_0) , and satisfy the relevant geophysical boundary conditions (see, e.g., [16, 19, 20]). Here $\Delta_H = \partial_{xx} + \partial_{yy}$ denotes the horizontal Laplacian operator. The global well-posedness (for all time and for all initial data) of strong solutions to the above three-dimensional system, subject to the relevant geophysical boundary conditions, has been proven first in [6] with full viscosity (i.e. $\nu_H > 0$ and $\nu_3 > 0$) and full diffusion (i.e. $\kappa_H > 0$ and $\kappa_3 > 0$), (see also, [14] for the case of Dirichlet boundary conditions). This result has been improved recently in [7] for the case of full viscosity (i.e. $\nu_H > 0$ and $\nu_3 > 0$) and only partial anisotropic vertical diffusion (i.e. $\kappa_H = 0$ and $\kappa_3 > 0$) which stands for the vertical eddy heat diffusivity turbulence mixing coefficient (see, e.g., [12], [13], for the geophysical justification). In the above results, the Coriolis forcing term, with rotation parameter R , did not play any role in proving the global regularity. This is contrary to the cases of the three-dimensional fast rotating Euler, Navier–Stokes and Boussinesq equations by [2, 3, 4, 5] where the authors take full advantage of the absence of resonances between the fast rotation and the nonlinear advection. This absence of resonances at the limit of fast rotation leads to strong dispersion and averaging mechanism (at the limit of fast rotation for large values R depending on the size of the initial data) that weakens the

nonlinear effects and hence allows for establishing the global regularity result in the viscous Navier-Stokes case, and prolongs the life-space of the solution in the Euler case (see also [8, 11] and references therein; in addition, see [1] for simple examples demonstrating the above mechanism).

In geophysical situations it is observed that the viscosity coefficients are very small, and that in fact $\nu_3 \ll \nu_H \ll 1$. Motivated by this observation and the above discussion it will become interesting to know of whether the inviscid (non-viscous) primitive equations are globally regular or that they develop singularity (blow up) in finite time. Since the rotation term did not play any role in establishing the global regularity in the viscous cases (cf. [6, 7]) one might as well ask the above question without the rotation term (i.e., consider the case with $R = 0$). Therefore, we will consider in this paper the inviscid primitive equations without the Coriolis rotation term, and we will show that for certain class of smooth initial data their corresponding smooth solutions will develop a singularity (blowup) in finite time.

In order to establish our result we will assume that we are given a smooth solution to the inviscid primitive equation. In section 2 we will derive a reduced equation that this smooth solution will satisfy. In section 3 we will follow [9] (see also [17]) to show that for certain class of initial data the corresponding solutions to this reduced equation blow up in finite time. In section 4 we provide a family of initial data whose corresponding smooth solutions to the inviscid primitive equations blow up in finite time. It is worth mentioning that similar approach has been introduced in [10] to show the blowup for the Prandtl equation of the boundary layer in the upper half-space.

2. DERIVATION OF A REDUCED EQUATION

In this section we will assume that we are given a conveniently smooth solution to the inviscid primitive equations. We will derive a reduced equation that this solution must satisfy by requiring that the solution fulfills certain symmetry conditions. Eventually, in section 3, we will show that for certain class of initial data the corresponding solutions of this reduced equation develop singularity in finite time.

First, let us consider the inviscid primitive equations without the Coriolis force:

$$u_t + uu_x + vu_y + wu_z + p_x = 0, \quad (6)$$

$$v_t + uv_x + vv_y + wv_z + p_y = 0, \quad (7)$$

$$p_z + T = 0, \quad (8)$$

$$T_t + uT_x + vT_y + wT_z = \kappa_H \Delta_H T + \kappa_3 T_{zz}, \quad (9)$$

$$u_x + v_y + w_z = 0, \quad (10)$$

in the horizontal channel $\Omega = \{(x, y, z) : 0 \leq z \leq H, (x, y) \in \mathbb{R}^2\}$; subject to the boundary conditions: no-normal flow and no heat flux in the vertical direction at the physical solid (top $z = H$ and bottom $z = 0$) boundaries, and periodic boundary conditions, say with period L , in horizontal directions. Observe that when the initial temperature $T_0 = 0$, then it is easy to see that any smooth solution to system (6)–(10), subject the above boundary conditions, must satisfy $T(x, y, z, t) \equiv 0$. Consequently, for any smooth solution with $T_0 = 0$ the velocity field satisfies the following system:

$$u_t + uu_x + vu_y + wu_z + p_x = 0, \quad (11)$$

$$v_t + uv_x + vv_y + wv_z + p_y = 0, \quad (12)$$

$$p_z = 0, \quad (13)$$

$$u_x + v_y + w_z = 0. \quad (14)$$

Since our goal is to establish the blowup for certain class of smooth solutions and initial data, we will further simplify matters and restrict ourselves to smooth solution to the above system that are independent of the y -variable and that the y -component of initial velocity vector field $v_0 = 0$. Once again it is clear that any smooth unique solution to the above system with these properties will satisfy $v(x, z, t) \equiv 0$, and that

u, w are functions of the spatial variables (x, z) and of the time t only. Consequently, we restrict ourselves further and only consider solutions that satisfy the following two-dimensional sub-system:

$$u_t + uu_x + wu_z + p_x = 0, \quad (15)$$

$$p_z = 0, \quad (16)$$

$$u_x + w_z = 0, \quad (17)$$

in the two-dimensional channel

$$M = \{(x, z) : 0 \leq z \leq H, x \in \mathbb{R}\},$$

subject to the no-normal flow on the vertical direction, and periodic boundary conditions in horizontal x -direction, namely,

$$w|_{z=H} = w|_{z=0} = 0, \quad (18)$$

and

$$u(x + L, z, t) = u(x, z, t); p(x + L, z, t) = p(x, z, t); w(x + L, z, t) = w(x, z, t). \quad (19)$$

Observe that the space of periodic functions with respect to x with the following symmetry

$$u(x, z, t) = -u(-x, z, t); p(x, z, t) = p(-x, z, t); w(x, z, t) = w(-x, z, t), \quad (20)$$

is invariant under the solution operator of system (15)–(17) subject to the boundary condition (18) and (19). Therefore, from now on we will restrict our discussion to the solutions of system (15)–(19) that satisfy the symmetry condition (20).

Let us denote by

$$\bar{\phi}(x, t) = \frac{1}{H} \int_0^H \phi(x, z, t) dz. \quad (21)$$

From equation (17) and the boundary condition (18), we have

$$0 = w(x, H, t) - w(x, 0, t) = \int_0^H w_z(x, z, t) dz = - \int_0^H u_x(x, z, t) dz. \quad (22)$$

Thus,

$$\bar{u}_x(x, t) = 0. \quad (23)$$

By differentiating equation (15), with respect to x , we obtain

$$\frac{\partial u_x}{\partial t} + (uu_x)_x + w_x u_z + w u_{zx} + p_{xx} = 0. \quad (24)$$

Equation (16) implies that $p(x, z, t)$ is only function of (x, t) , i.e., $p(x, z, t) = p(x, t)$. By averaging (24) with respect to the z variable over $[0, H]$ and using (23), we obtain

$$\overline{(uu_x)_x} + \overline{w_x u_z} + \overline{w u_{zx}} + p_{xx} = 0. \quad (25)$$

Notice that (18) implies

$$w_x(x, H, t) = w_x(x, 0, t) = 0. \quad (26)$$

Thanks to (17) and (26), integrating by parts gives

$$\begin{aligned} \overline{w_x u_z} + \overline{w u_{zx}} &= -\overline{w_{xz} u} - \overline{w_z u_x} \\ &= \overline{u_{xx} u} + \overline{(u_x)^2} = \overline{(uu_x)_x}. \end{aligned}$$

As a result, (25) can be reduced to

$$2\overline{(uu_x)_x} + p_{xx} = 0. \quad (27)$$

Thus,

$$2\overline{uu_x} + p_x = C(t), \quad (28)$$

for some function $C(t)$. Since p_x and $\overline{uu_x}$ are odd functions, with respect to the variable x (see condition (20)), then

$$\overline{uu_x}(0, t) = p_x(0, t) = 0.$$

Therefore,

$$C(t) = 2\overline{uu_x}(0, t) + p_x(0, t) = 0, \quad (29)$$

and consequently,

$$p_x = -2\overline{uu_x}. \quad (30)$$

Substituting (30) into system (15)–(17) we obtain the closed system

$$\frac{\partial u}{\partial t} + uu_x + wu_z - 2\overline{uu_x} = 0, \quad (31)$$

$$u_x + w_z = 0, \quad (32)$$

subject to the boundary conditions (18)–(20). In particular, by differentiating with respect to x we have

$$\frac{\partial u_x}{\partial t} + (uu_x)_x + w_x u_z + w u_{xz} - 2(\overline{uu_x})_x = 0. \quad (33)$$

Let us consider the restriction of the evolution of equation (33) on the line $x = 0$. Since $u(x, z, t)$ is an odd-function, and $w(x, z, t)$ is an even-function, with respect to the variable x we have:

$$u(0, z, t) = 0; w_x(0, z, t) = 0.$$

This together with (33) imply

$$u_{tx}(0, z, t) + (u_x(0, z, t))^2 + w(0, z, t)u_{xz}(0, z, t) - 2(\overline{u_x(0, z, t)})^2 = 0. \quad (34)$$

Recalling that $w_z(0, z, t) = -u_x(0, z, t)$, and denoting by $W(z, t) = w(0, z, t)$, we obtain

$$W_{tz} - (W_z)^2 + WW_{zz} + \frac{2}{H} \int_0^H (W_z)^2 dz = 0, \quad (35)$$

with the boundary conditions:

$$W(0, t) = W(H, t) = 0. \quad (36)$$

This is a closed equation that we will investigate in section 3.

3. SELF-SIMILAR BLOWUP OF THE REDUCED EQUATION

In this section we will investigate the blowup of the nonlinear integro-differential boundary value problem (35)–(36). The analytical and numerical investigation of this type of equations have been the subject matter of the papers [17, 18]; see also [15] for additional new results. The presentation below follows [9] (see also [17] section 4, and references therein), where the same problem, arising in a different fluid dynamical context, has been investigated. For the sake of completeness we provide below the full details.

Noting that equation (35) is invariant for the scaling

$$W(z, t) \mapsto \lambda W(z, \lambda t),$$

therefore, we look for a self-similar solution in the form

$$W(z, t) = \frac{\varphi(z)}{1-t}, \text{ with } \varphi(0) = \varphi(H) = 0.$$

This solution starts from the initial profile $W(z, 0) = \varphi(z)$, and blows up at time $t = 1$ at every $z \in (0, H)$, for which $\varphi(z) \neq 0$. Therefore, the idea is to construct a nontrivial initial profile $\varphi(z)$. Using the above ansatz and substituting in (35) we obtain the following reduced nonlinear nonlocal boundary value problem:

$$\varphi' - (\varphi')^2 + \varphi\varphi'' + \frac{2}{H} \int_0^H (\varphi'(z))^2 dz = 0, \quad \varphi(0) = \varphi(H) = 0, \quad (37)$$

that we need to show that it has a nontrivial solution φ .

Let $m > 0$ be a given free parameter, we consider instead of (37) the nonlinear boundary value problem

$$\varphi' - (\varphi')^2 + \varphi\varphi'' + m^2 = 0, \quad \varphi(0) = \varphi(H) = 0. \quad (38)$$

We observe that in case the boundary value problem (38) has a solution, which we denote by φ_m , then φ_m is nontrivial because $m > 0$. Moreover, integrating equation (38) over $0 < z < H$ yields

$$m^2 = \frac{2}{H} \int_0^H (\varphi'_m(z))^2 dz.$$

Consequently, φ_m is also a nontrivial solution to the original nonlocal boundary value problem (37). Therefore, we will focus now on showing that problem (38) has a nontrivial solution for every $m > 0$ given.

First we look for a positive solution φ to the boundary value problem (38). Denoting by $\psi := \varphi'$, then any local smooth solution to the second order ordinary differential equation given in (38) can be parameterized by an integral curve $(\varphi(z), \psi(z))$, in the phase portrait space (φ, ψ) . Suppose that we can parameterize, say locally, the integral curve in terms of ψ instead of z , i.e., that we can invert the relationship between ψ and z and express z as function ψ . Then equation (38) becomes

$$\frac{1}{\varphi} \frac{d\varphi}{d\psi} = \frac{\psi}{\psi^2 - \psi - m^2} = \frac{\psi}{(\psi - \psi_+)(\psi - \psi_-)},$$

where ψ_{\pm} are the singular points

$$\psi_+(m) := \sqrt{m^2 + 1/4} + 1/2 > 0, \quad \psi_-(m) := -\sqrt{m^2 + 1/4} + 1/2 < 0.$$

Integrating the above equation we obtain general integral curves in the phase portrait for (φ, ψ)

$$\varphi = C |\psi - \psi_+|^{\frac{\psi_+}{\psi_+ - \psi_-}} |\psi - \psi_-|^{\frac{-\psi_-}{\psi_+ - \psi_-}}, \quad (39)$$

with an integration constant C . Since we are interested in positive solutions φ we take $C > 0$. In order for this curve to be a solution to the boundary value problem (38), and that it can be parameterized in terms of $z \in [0, H]$, one has to require that for each $C > 0$ the above curve yields a solution satisfying

$$(\varphi, \psi)(z = 0) = (0, \psi_+), \quad (\varphi, \psi)(z = H) = (0, \psi_-). \quad (40)$$

Injecting (39) into (38) yields

$$\frac{d\psi}{dz} = \frac{-1}{C} |\psi - \psi_+|^{\frac{-\psi_-}{\psi_+ - \psi_-}} |\psi - \psi_-|^{\frac{\psi_+}{\psi_+ - \psi_-}}, \quad (41)$$

which shows that ψ is a decreasing function and hence the inversion between the variables z and ψ is valid over the interval $\psi_- \leq \psi \leq \psi_+$. In particular, for every $\psi \in [\psi_-, \psi_+]$ and by virtue of (41) and (40) we have

$$z(\psi) = \int_{\psi_+}^{\psi} \frac{dz}{d\psi} d\psi = \int_{\psi_+}^{\psi} \frac{\varphi}{\psi + \psi^2 - m^2} d\psi = -C \int_{\psi_+}^{\psi} |\psi - \psi_+|^{\frac{\psi_-}{\psi_+ - \psi_-}} |\psi - \psi_-|^{\frac{-\psi_+}{\psi_+ - \psi_-}} d\psi. \quad (42)$$

Moreover, from the other boundary condition in (40) the length of interval is determined by

$$\begin{aligned}
H = z(\psi_-) &= \int_{\psi_+}^{\psi_-} \frac{dz}{d\psi} d\psi = \int_{\psi_+}^{\psi_-} \frac{\varphi}{\psi + \psi^2 - m^2} d\psi \\
&= C \int_{\psi_-}^{\psi_+} |\psi - \psi_+|^{\frac{\psi_-}{\psi_+ - \psi_-}} |\psi - \psi_-|^{\frac{-\psi_+}{\psi_+ - \psi_-}} d\psi \\
&= \frac{C}{\psi_+ - \psi_-} B\left(\frac{\psi_+}{\psi_+ - \psi_-}, \frac{-\psi_-}{\psi_+ - \psi_-}\right) \\
&= \frac{C}{2\sqrt{m^2 + 1/4}} B\left(\frac{1}{2} + \frac{1}{4\sqrt{m^2 + 1/4}}, \frac{1}{2} - \frac{1}{4\sqrt{m^2 + 1/4}}\right),
\end{aligned} \tag{43}$$

where $B(\cdot, \cdot)$ is the Beta function.

Hence with the given $H > 0$, for each $m > 0$ there is a unique $C > 0$, depending on H and m , such that (43) holds.

From all the above we concluded that there is a solution φ of (38) satisfying

$$\varphi(0) = 0 = \varphi(H), \quad \varphi > 0 \text{ on } 0 < z < H, \quad \text{and} \quad \frac{2}{H} \int_0^H (\varphi'(z))^2 dz = m^2.$$

This in turn also shows that φ is a nontrivial solution to the nonlinear nonlocal boundary value problem (37).

From (41), (40) and the above discussion one also concludes that

$$\varphi_{zz}(0) = 0 = \varphi_{zz}(H). \tag{44}$$

Furthermore, the symmetry of equation (38) implies that $-\varphi(H - z)$ is a negative solution to (38).

In conclusion, we have obtained a one-parameter family of blowup solutions of (35)-(36), which blow up at every $z \in (0, H)$, as $t \rightarrow 1^-$.

Remark 1. If we choose the constant C in (43) such that $z(\psi_-) = \frac{H}{2}$, then condition (44) holds at $z = \frac{H}{2}$ instead of $z = H$. This in turn will allow us to glue the positive solution defined on the interval $[0, \frac{H}{2}]$, and which is also satisfying (40) $z = \frac{H}{2}$ instead of $z = H$, with its negative counterpart at $z = \frac{H}{2}$. Hence, this idea will allow us construct sign-changing blowup solutions. In fact, this is the type of solutions that are constructed in [9] and [17]. Of course one can repeat the above and get profiles that change signs as many times as one wants.

4. BLOWUP OF THE SMOOTH SOLUTIONS TO THE PRIMITIVE EQUATIONS

In this section we will demonstrate our main result and show that for certain class of initial data the corresponding smooth solutions to the primitive equations (15)-(17), subject to the boundary conditions (18)-(20), blow up in finite time. First, let us prove the following proposition concerning the uniqueness of solutions to equation (35)-(36).

Proposition 1. *Let W_1, W_2 be two solutions of (35)-(36) which are in $L^2((0, T); H^2)$, with $W_1(z, 0) = W_2(z, 0)$. Then $W_1(z, t) = W_2(z, t)$, for all $t \in [0, T)$. In particular, $W(z, t) = \frac{\varphi(z)}{1-t}$ is the only solution of (35)-(36) in the space $L^2((0, T); H^2)$, for all $T \in [0, 1)$, with initial data $\varphi(z)$, where $\varphi(z)$ is any nontrivial solution of the boundary value problem (37) that was established in section 3.*

Proof. Let $V = W_1 - W_2$, and $\overline{W} = \frac{1}{2}(W_1 + W_2)$. Then (35)-(36) imply:

$$V_{tz} - 2\overline{W}_z V_z + V\overline{W}_{zz} + \overline{W}V_{zz} + \frac{4}{H} \int_0^H \overline{W}_z V_z dz = 0, \tag{45}$$

with the boundary conditions:

$$V(0, t) = V(H, t) = 0. \quad (46)$$

Multiplying (45), integrating with respect to z over $[0, H]$, integrating by parts and using the boundary conditions (36) and (46) gives:

$$\frac{1}{2} \frac{d}{dt} \int_0^H V^2(z, t) dz = \frac{5}{2} \int_0^H \overline{W}_z (V_z)^2 dz - \int_0^H \overline{W}_{zz} V V_z dz.$$

Thus, by Hölder inequality we obtain

$$\frac{d}{dt} \|V_z(t)\|_{L^2(0, H)}^2 \leq 5 \|\overline{W}_z\|_{L^\infty(0, H)} \|V_z(t)\|_{L^2(0, H)}^2 + 2 \|\overline{W}\|_{H^2(0, H)} \|V(t)\|_{L^\infty(0, H)} \|V_z(t)\|_{L^2(0, H)}.$$

By virtue of the one-dimensional Sobolev imbedding theorem, and thanks to the boundary conditions (46) one can apply the Poincaré inequality, to obtain:

$$\frac{d}{dt} \|V_z(t)\|_{L^2(0, H)}^2 \leq c \|\overline{W}\|_{H^2(0, H)} \|V_z(t)\|_{L^2(0, H)}^2.$$

Applying Gronwall's inequality we then conclude that $\|V_z(\cdot, t)\|_{L^2(0, H)} = 0$, for all $t \in [0, T]$. Again, thanks to (46) we apply Poincaré inequality to conclude the uniqueness part of the statement of the proposition. The second part of the statement is an immediate corollary of the first part. \square

Theorem 2. *Let $\varphi(z)$ be any nontrivial solution of the boundary value problem (37), and let k be any positive integer. Suppose that $(u(x, z, t), w(x, z, t))$ is a smooth solution of (15)-(17), subject to the boundary conditions (18)-(20), with initial value*

$$u_0(x, z) = \frac{-L}{2\pi k} \sin\left(\frac{2\pi k}{L}x\right) \varphi'(z) \quad \text{and} \quad w_0(x, z) = \cos\left(\frac{2\pi k}{L}x\right) \varphi(z).$$

Then this solution blows up in finite time.

Proof. Based on the derivation in section 2 the function $w(0, z, t)$ satisfies (35)-(36), with initial value $w(0, z, 0) = \varphi(z)$. Therefore, by Proposition 1 one concludes that for as long as the solution $w(x, z, t)$ exists and is smooth must satisfies $w(0, z, t) = \frac{\varphi(z)}{1-t}$. Consequently, $w(x, z, t)$ must lose regularity at sometime in the interval $t \in [0, 1]$. \square

5. ACKNOWLEDGMENTS

The authors would like to thank the Banff International Research Station (BIRS), Canada, for the kind and warm hospitality where part of this work was completed. The work of C. Cao is supported in part by the NSF grant no. DMS-1109022. S. Ibrahim is partially supported by NSERC #371637-2009 grant. E.S. Titi is supported in part by the NSF grants DMS-1009950, DMS-1109640, and DMS-1109645. E.S. Titi also acknowledges the support of the Alexander von Humboldt Stiftung/Foundation and the Minerva Stiftung/Foundation.

REFERENCES

- [1] A. Babin, A.A. Ilyin, and E.S. Titi, *On the regularization mechanism for the spatially periodic Korteweg–de Vries equation*, Communications in Pure and Applied Mathematics, **64** (2011), 591–648.
- [2] A. Babin, A. Mahalov and B. Nicolaenko, *Regularity and integrability of 3D Euler and Navier–Stokes equations for rotating fluids*, Asymptot. Anal. **15:2** (1997), 103–150.
- [3] A. Babin, A. Mahalov and B. Nicolaenko, *Global regularity of 3D rotating Navier–Stokes equations for resonant domains*, Indiana Univ. Math. J. **48:3** (1999), 1133–1176.
- [4] A. Babin, A. Mahalov and B. Nicolaenko, *On the regularity of three-dimensional rotating Euler–Boussinesq equations*, Mathematical Models and Methods in Applied Sciences. **9:7** (1999), 1089–1121.

- [5] A. Babin, A. Mahalov and B. Nicolaenko, *Fast singular oscillating limits and global regularity for the 3D primitive equations of geophysics*, Math. Model. Numer. Anal. **34** (2000), 201–222.
- [6] C. Cao and E.S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. **166** (2007), 245–267.
- [7] C. Cao and E.S. Titi, *Global well-posedness of the three-dimensional stratified primitive equations with partial vertical mixing turbulence diffusion*, Comm. Math. Phys. **310** (2012), 537–568.
- [8] J.-Y. Chemin, B. Desjardines, I. Gallagher and E. Grenier, *Mathematical Geophysics. An introduction to rotating fluids and the Navier-Stokes Equations*, Oxford lecture series in Mathematics and Its Applications, Vol. 32. Clarendon Press, Oxford, 2006.
- [9] S. Childress, G.R. Ierley, E.A. Spiegel and W.R. Young, *Blow-up of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation-point form*, J. Fluid Mech. **203** (1989), 1–22.
- [10] W. E and B. Engquist, *Blowup of solutions of the unsteady Prandtl's equation*, Comm. Pure Appl. Math. **50** (1997), 1287–1293.
- [11] P.F. Embid and A. J. Majda, *Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity*, Comm. Partial Diff. Eqs. **21** (1996), 619–658.
- [12] A.E. Gargett, *Vertical eddy diffusivity in the ocean interior*, J. Mar. Res., **42** (1984), 359–393.
- [13] C.J.R. Garrett, *Oceanography: A stirring tale of mixing*, Nature, **364** (1993), 670–671.
- [14] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity **20** (2007), 2739–2753.
- [15] S. Ibrahim, K. Nakanishi and E.S.Titi, *Global regularity vs. blowup dichotomy for one-dimensional models arising from fluid dynamics*, Preprint.
- [16] A. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Institute Lecture Notes No. 9 (American Mathematical Society, Providence, RI, 2003).
- [17] H. Okamoto, *Well-posedness of the generalized Proudman-Johnson equation without viscosity*, Journal of Mathematical Fluid Mechanics **11** (2009), 46–59.
- [18] H. Okamoto and J. Zhu, *Some similarity solutions of the Navier-Stokes equations and related topics*, Taiwanese Journal of Mathematics **4:1** (2000), 65–103.
- [19] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, New York, 1987.
- [20] R. Salmon, *Lectures on Geophysical Fluid Dynamics*, Oxford University Press, New York, Oxford, 1998.

(C. Cao) DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, UNIVERSITY PARK, MIAMI, FL 33199, USA.

E-mail address: caoc@fiu.edu

(S. Ibrahim) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VICTORIA, PO BOX 3060, STN CSC, VICTORIA, BC V8W 3R4, CANADA

E-mail address: ibrahims@uvic.ca

(K. Nakanishi) DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: n-kenji@math.kyoto-u.ac.jp

(E.S. Titi) DEPARTMENT OF COMPUTER SCIENCE AND APPLIED MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL. **Also:** DEPARTMENT OF MATHEMATICS, AND DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING, UNIVERSITY OF CALIFORNIA, IRVINE, CA 92697-3875, USA. FELLOW OF THE CENTER OF SMART INTERFACES, TECHNISCHE UNIVERSITÄT DARMSTADT, GERMANY.

E-mail address: etiti@math.uci.edu

E-mail address: edriss.titi@weizmann.ac.il